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Dynamic Graphs, Networks-of-Networks with Faulty Inter-network Connections, Pinning-control Synchronization Coordinator: Coincidences and Confluences

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Abstract

Investigation of large network-of-networks system structures is an appealing timely research topic, in particular when of permanent and recoverable faults may occur in coupling node connections within different networks. For the modeling of such a multi-network structure, in this paper, the coupling terms in the same network and those among different networks are described separately in order to distinguish clearly the multi-network feature. A dimension-transformation matrix is used where needed to deal with the mismatched dimensions of nodes in different networks. A synchronization decentralized control infrastructure beyond traditional feedback is designed via the theory of pinning control schemes. A sufficient stability condition under the pinning control is derived that guarantees convergence to a steady-state operating synchronization while overall multi-network structure remains technically stable. Simulation results for a bench-mark case-study of three coupled networks of node systems of notoriously complex nonlinear dynamics demonstrate control strategy employing pinning control theory can enforce stabilized synchronization despite any faults.

Keywords: Complex dynamic networks, Chaos dynamics, Decentralized node controls, Controlled synchronization, Faults, Interconnection edges, Networks of dynamic networks, Pinning control coordinator, Supervisory control beyond feedback

Introduction

In their recent article *Wang et al., (2022)* rightly argued that a new era begins focused on exploration of the symbiotic autonomous systems *(Wang Y, et al., 2018)*, [1] and Complex Multi-Network Systems (CMNS; [2,3] from synergies of Artificial Intelligence (AI) advances and some human-intelligence-in-the-loop beyond the traditional feedback [4,5]. Indeed, a new scientific investigation era of networked systemic structures has emerged [6,7] following the discoveries of small-worlds of [8] and scale-free networks of [9], both being advanced follow up developments built upon random graphs [10] of [11] and the early graph dynamics [12] of *Aizerman, et al.,* [13]. Ever since studies of complex dynamical networks do expand via new discoveries in complex network-like systemic structures [14-22]. And so do the application-oriented investiga-

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tions of complex dynamic network-like systems such as artificial neural networks, brain neuronal networks, communication and computer networks, electric power grids, social networks, etc. to name a few. Nowadays, it is well known the functionality of complex dynamic networks depend heavily on both convergence to stable operating equilibrium and synchronization dynamics. Therefore, about two decades both controlled stabilization and synchronization became the main research endeavors world-wide and considerable advances in the applications and the theory and of complex networks have been achieved [18,23]. In particular, idea and concept of pinning control of *Chen, et al.,* [24-27] appeared amazingly effective and yielded a research direction with impressive results (e.g., see [6,18,28-32]).

Unlike the topology of a single network (e.g., see [26,28,30,33,34] and references therein), the systemic structure of networks of coupled networks does have a great influence on overall dynamics (e.g., see [35-40] and references therein). The change of the node connecting edge between any two networks will have a great impact on the stability of the whole network, thus affecting overall network stability and synchronization too. Due to the interaction among networks, the failure of a node in the network will not only affect its own network, but also lead to a kind of chain reaction, resulting in multiple network failure. In real-world networks, after some time of the self-adjustment (e.g. biological network has the ability to self-recovery) or the certain artificial intervention (e.g. traffic network congestion will be diverted by traffic police), some networks coupling connection failures can return to normal working conditions. Therefore, the inter-network faults can be either permanent and/or recoverable faults. Thus, studies of the problem of multi-network synchronization control in the presence of both recoverable and permanent faults among networks, operating within a multi-network, have profound importance [37,38,41-43]. Studies of clustering synchronization control in complex networks are already fairly mature now. On the other hand, it should be noted controlled multi-network synchronization and cluster synchronization involve similar underlying issues [38,44]. Thus, the study of multi-network can be combined with the characteristics of multi-network model coupling and learned from the cluster synchronization research models as well as the respective methods [44,45].

In particular, the pinning controller is designed, and the gain of the controller is adjusted, the clustering synchronization of complex network is realized in work [37]. But the coupling strength of the network appears large, which does not seem realistic. In [2], the coupling control strategy is adopted to adjust the coupling strength by means of adaptation, and clustering synchronization of network can be realized under a relatively weak coupling. In [6], based on the pinning control scheme, the clustering synchronization of network is realized by the intermittent control method. Due to influence from the environment, human factors inclusive, the coupling strength of nodes in a complex network may change randomly with time and so can inter-network connections [37]. Recoverable failure that is the random and unpredictable is due to interference or attack among networks. In [43], random variables are used to describe the random variation of coupling strength, a class of neural network synchronization control problem is studied. In [35], the problem of complex network synchronization with nonlinear stochastic distribution is studied. The stochastic variation of nodes is represented by introducing the random variables with Bernoulli distribution, which seems reasonable; however, the connections among nodes are not considered. In [36], the cascade failures caused by the attack on complex network are studied; investigation is rightly carried out from the occurrence to the propagation of the process underlying the evolution dynamics [37]. But it is not studied from the control point and does not consider the recoverability of node connection failures. In [42], the complex dynamic network in the case of fault is studied. The nodes in the network are randomly switched between all connections and failures that possibly may occur. The failure time ratio is designed to synchronize the network.

In the existing literature, not many researches do report on the multi-network settings, i.e. the network-of-networks, of the operating synchronization when inter-connection failures occur [45]. It appears as if the present literature does not consider the coupling interconnection edge may be subject to deliberate interference, failure, or even attacked hence concept of connective stability [25] seems ignored. Thus investigating categories of recoverability and/or permanent faults among nodes of different sub-networks within multi-networks seems timely and thus of considerable importance. However, the issue of controlled synchronization of multi-networks to a stable operating equilibrium, unlike that of networked control systems [32], is much more involved [25,45]. Hence complex networks and multi-networks are to be driven to the stable operating equilibrium in a similar way as the recurrent neural networks with time-varying delays need [24,46].

In this paper, the multi-network mathematical model is used to study the synchronization control problem in network-of-networks by considering both the recoverable and the permanent faults of the interconnection and possibly the inter-network connecting link. Bernoulli-distribution probabilistic variable is employed to capture the change events between failure and recovery or vice-versa. On the theoretical grounds of pinning control [3,33], sufficient conditions for controlled synchronization within respective multi-network structure are derived here. By relying on concepts and ideas of Lyapunov stability [5], sufficient conditions for synchronization within the respective network are found. These derived results are verified to be correct by means of simulation experiments on a case-study network-of-networks example comprising a four-node Lorenz hyper-chaotic sub-network as well as one Rossler and one Chen sub-networks both having three nodes.

Further, this paper is written as follows. In Section II, there are explored the system models for multi-networks with two types of inter-network failures as well as certain mathematical preliminaries. Section III presents the main results on pinning control design for network synchronization and its asymptotic property. These results are verified in Section IV by means of numerical and simulation results for a complex multi-network case-study. Then concluding remarks and references follow.

On Networks-of-Networks and Inter-Network Connection Faults

The underlying systems theory requires certain preliminary mathematical modeling for networks of complex networks as well as some specific mathematical issues to be addressed first. Then exploration of multi-network systemic structures and dynamic graphs with inter-network faults or failures can be carried out to the full and can reveal essential differences in comparison with standard dynamic networks.

On Mathematical Models of Network-of-Networks with Inter-network Connection Faults

Let consider r dynamical networks and suppose the k-th network is composed N_k of nodes whose associated dynamic systems have dimensions n_k. Also, networks where both *permanent* and *recoverable* faults of coupled nodes among different networks may occur are encompassed in this study. Thus the i-th node in the k-th network are modeled by:

$$\begin{cases} \dot{x}_i^k(t) = f^k(x_i^k(t)) + \sum_{j=1, j\neq i}^{N_k} a_{ij}^{kk}(x_j^k(t) - x_i^k(t)) \\ \quad + \sum_{l=1, l\neq k}^{r} \sum_{j=1}^{[\alpha N_l]} (m_{ij}^{kl}(t) + a_{ij}^{kl})(\Gamma^{kl} x_j^l(t) - x_i^k(t)) \\ \quad + \sum_{l=1, l\neq k}^{r} \sum_{j=[\alpha N_l]+1}^{N_l} \delta_{ij}^{kl}(t) a_{ij}^{kl}(\Gamma^{kl} x_j^l(t) - x_i^k(t)), \\ \quad i = 1, 2, \ldots, N_k, \ k = 1, 2, \ldots, r. \end{cases} \tag{1}$$

Quantities in here, denote: $t \in \mathbb{R}^+$ is the independent variable of time; $x_i^k(t) = (x_{i1}^k(t), x_{i2}^k(t), \cdots, x_{in_k}^k(t))^{\mathrm{T}} \in \mathbb{R}^{n_k}$ is the state vector of node i in the k-th network; $f^k(\cdot): \mathbb{R}^{n_k} \to \mathbb{R}^{n_k}$ is a nonlinear vector-valued function of a given node, describing functional dynamics of nodes. Matrix $A_{kk} = (a_{ij}^{kk}) \in \mathbb{R}^{N_k \times N_k}$ describes the coupling configuration, representing topological structure and coupling strength of the k-th network, where $a_{ij}^{kk} > 0$ if there is a healthy connection edge between nodes 'i' and 'j' $(i \neq j)$ and $a_{ij}^{kk} = 0$ if otherwise; $A_{kl} = (a_{ij}^{kl}) \in \mathbb{R}^{N_k \times N_l}$ is the matrix of external connections. Real-valued constant α represents the proportion of nodes where failures occurrence $[\alpha N_l]$ represents the number of nodes in which *permanent* faults occur, while $N_l - [\alpha N_l]$ represents the number of nodes in which *recoverable* faults occur. Variable $m_{ij}^{kl}(t)$ represents the fault signal pointing to connection problem between the l-th and the k-th sub-network. Quantity Γ^{kl} is a chosen suitable dimension-transformation matrix, highlighted in the sequel.

Further, a random variable $\delta_{ij}(t)$ is employed to describe the probabilistic events of connection failure and recovery in considered network systemic structure (1), namely:

$$\delta_{ij}^{kl}(t) = \begin{cases} 1 & \text{Connection edge is operational,} \\ 0 & \text{Connection edge is faulty or failed.} \end{cases} \tag{2}$$

Assumption 1 [45]: Probabilistic events of connection failure or malfunctioning $\delta_{ij}^{kl}(t)$ obeys the Bernoulli distribution hence the equalities $\mathrm{Prob}\{\delta_{ij}^{kl}(t) = 1\} = \bar{\delta}_{ij}^{kl}$, $\mathrm{Prob}\{\delta_{ij}^{kl}(t) = 0\} = 1 - \bar{\delta}_{ij}^{kl}$, where $\bar{\delta}_{ij}^{kl}$ is the probability that the connection will not fail, are satisfied.

Suppose that connection edge of network (1) is represented as follows:

$$a_{ii}^{kk} = -\sum_{j=1, j\neq i}^{N_k} a_{ij}^{kk} - \sum_{l=1, l\neq k}^{r} \sum_{j=1}^{[\alpha N_l]} (m_{ij}^{kl}(t) + a_{ij}^{kl}) - \sum_{l=1, l\neq k}^{r} \sum_{j=[\alpha N_l]+1}^{N_l} \delta_{ij}^{kl}(t) a_{ij}^{kl}, \ i = 1, 2\ldots, N_k, k = 1, \ldots, r. \tag{3}$$

Then, network (1) can be described [31,45] a

$$\begin{cases} \dot{x}_i^k(t) = f^k(x_i^k(t)) + \sum_{j=1}^{N_k} a_{ij}^{kk} x_j^k(t) \\ \quad + \sum_{l=1, l\neq k}^{r} \sum_{j=1}^{[\alpha N_l]} (m_{ij}^{kl}(t) + a_{ij}^{kl}) \Gamma^{kl} x_j^l(t) \\ \quad + \sum_{l=1, l\neq k}^{r} \sum_{j=[\alpha N_l]+1}^{N_l} \delta_{ij}^{kl}(t) a_{ij}^{kl} \Gamma^{kl} x_j^l(t) \\ i = 1, 2, \ldots, N_k, k = 1, 2, \ldots, r. \end{cases} \tag{4}$$

Mathematical Preliminaries: Definitions and Assumptions

Certain assumptions and definitions are in order to present.

Definition 1 [39,45]: If the expectation property

$$\lim_{t\to\infty} E\left\{\sum_{i=1}^{N_k} \| x_i^k(t) - s^k(t) \|^2 \right\} = 0, \quad i = 1, \ldots, N_k, k = 1, 2, \ldots, r, \tag{5}$$

where $s^k(t)$ represents a solution for an isolated node system in the k-th network and applies for each of the multi-network nodes, then the multi-network system dynamics (4) is said to be *asymptotically Lyapunov stable in the mean-square sense.*

In Definition 1, it is important to note the solution of an isolated node

$$\dot{s}^k(t) = f^k(s^k(t)) \text{ and } s^k \in \mathbb{R}_0^{+\infty} \to \mathbb{R}^n \tag{6}$$

may represent any kind of *steady-state operation*. That is, (6) may be an *equilibrium state, a regular periodic orbit, a chaotic orbit, or a trajectory converging to a random steady-state evolution* that may not have analytical description [8].

Definition 2: Suppose $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, $B = (b_{ij}) \in \mathbb{R}^{p\times q}$. Then the following Kronecker product matrix is obtained

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \in \mathbb{R}^{mp\times nq}, \tag{7}$$

which satisfies essential properties:

(i) $(A + B) \otimes C = A \otimes C + B \otimes C$;

(ii) $(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$.

Assumption 2 [45]: There exists a constant $\theta_k > 0$, such that the nonlinear function $f^k(\cdot)$ of the k-th node dynamics satisfies

$$(x - y)^{\mathrm{T}}(f^k(x) - f^k(y)) \leq \theta_k (x - y)^{\mathrm{T}}(x - y), \quad \forall x, y \in \mathbb{R}^{n_k} \tag{8}$$

Assumption 3 [45]: Changes in coupling connection edges are assumed to be bounded, and solely occurrence of possible coupling faults among different networks are considered.

Assumption 4 [45]: Interconnections $0 < \underline{m}_{ij}^{kl} \le m_{ij}^{kl}(t) \le \overline{m}_{ij}^{kl}$ are assumed to have edges $m_{ij}^{kl}(t)$ that satisfy update rates

$$\dot{m}_{ij}^{kl}(t) = -\beta e_i^k(t)^{\mathrm{T}} \Gamma^{kl} s^l(t), \tag{9}$$

with $i = 1,2,\ldots,N_k$, $j = 1,2,\ldots,N_l$, $k,l = 1,2,\ldots,r$, where $\underline{m}_{ij}^{kl}$ and $\overline{m}_{ij}^{kl}$ are the upper and lower bounds of $m_{ij}^{kl}(t)$ while β is a positive constant.

Assumption 5 [45]: Suppose sets $V = \{i_1, i_2, \ldots, i_N\}$ and $V_{pin} = \{i_1, i_2, \ldots, i_l\}$ are defined such as to constitute the set the *total* nodes and the selected *pinned* nodes, respectively, for the controlled multi-network (3). Thus all nodes in $V \setminus V_{pin}$ are accessible from the pinned node set V_{pin}; i.e., for any node $i \in V \setminus V_{pin}$, always a node $j \in V_{pin}$ can be found, such that there is a functional path from node 'j' to node 'i'.

Assumption 6 [45]: When a node has a recoverable fault, the external coupling connected to the node is assumed disconnected, but the connection is resumed at the same time, thus $\delta_{i1}^{kl}(t) = \delta_{i2}^{kl}(t) = \cdots = \delta_{iN}^{kl}(t) = \delta_i^{kl}(t),\ i = 1,2,\ldots,N$, while the internal couplings are assumed not affected. In here, $\delta_i^{kl}(t)$ is also assumed to obey Bernoulli distribution and to satisfy $\mathrm{Prob}\{\delta_i^{kl}(t) = 1\} = \bar{\delta}_i^{kl},\ \mathrm{Prob}\{\delta_i^{kl}(t) = 0\} = 1\text{-}\bar{\delta}_i^{kl}$ (where $\bar{\delta}_i^{kl}$ is the probability that the connection will not fail).

Assumption 7 [45]: There exist non-negative bounded functions $s^k(t)$, $s^k(t)$, $s^k(t)$.

Error Dynamics Under a Supervisory Control Design

Assumptions 4 and 5 enable to re-write the multi-network system model (4) as follows:

$$\begin{cases} \dot{x}_i^k(t) = f^k(x_i^k(t)) + \sum_{j=1}^{N_k} a_{ij}^{kk} x_j^k(t) \\ \quad + \sum_{l=1,l\ne k}^{r} \sum_{j=1}^{[\alpha N_l]} (m_{ij}^{kl}(t) + a_{ij}^{kl}) \Gamma^{kl} x_j^l(t) \\ \quad + \sum_{l=1,l\ne k}^{r} \sum_{j=[\alpha N_l]+1}^{N_l} \delta_i^{kl}(t) a_{ij}^{kl} \Gamma^{kl} x_j^l(t) \\ i = 1,2,\ldots,N_k, k = 1,2,\ldots,r. \end{cases} \tag{10}$$

The supervisory synchronization controller to be designed is assumed under the principles of pinning control theory of Guanrong Chen [28,46]. Therefore the i -th node in the k -th network is described by means of the equations

$$\begin{cases} \dot{x}_i^k(t) = f^k(x_i^k(t)) + \sum_{j=1}^{N_k} a_{ij}^{kk} x_j^k(t) \\ \quad + \sum_{l=1,l\ne k}^{r} \sum_{j=1}^{[\alpha N_l]} (m_{ij}^{kl}(t) + a_{ij}^{kl}) \Gamma^{kl} x_j^l(t) \\ \quad + \sum_{l=1,l\ne k}^{r} \sum_{j=[\alpha N_l]+1}^{N_l} \delta_i^{kl}(t) a_{ij}^{kl} \Gamma^{kl} x_j^l(t) + u_i^k(t), \\ i = 1,2,\ldots,N_k,\ k = 1,2,\ldots,r, \end{cases} \tag{11}$$

Inspired by the theory of pinning control schemes [24,28,31,46] the class of controllers

$$\begin{cases} u_i^k(t) = -\sum_{j=1}^{N_k} h_{ij}^{kk}(t) s^k(t) - \sum_{j=[\alpha N_l]+1}^{N_l} h_{ij}^{kl} \Gamma^{kl} s^l(t) - d_i^k e_i^k(t), \\ \dot{h}_{ij}^{kk}(t) = e_i^k(t)^{\mathrm{T}} s^k(t),\ \dot{h}_{ij}^{kl}(t) = e_i^k(t)^{\mathrm{T}} \Gamma^{kl} s^l(t), \quad i \in \phi_k, \\ u_i^k(t) = -d_i^k e_i^k(t), \quad i = \tilde{\phi}_k - \phi_k, \end{cases} \tag{12 a}$$

are taken into consideration [45] where: $h_{ij}^{kk}(t)$ and $h_{ij}^{kl}(t)$ are some one-dimensional variables while represent feedback control gains that satisfy rule

$$\begin{cases} d_i^k > 0, i = 1,\ldots,l_k, \\ d_i^k = 0, i = l_k + 1,\ldots,N_k. \end{cases} \tag{12 b}$$

Furthermore, for dynamic networks and multi-networks it is proposed to distinguish the concepts of an *inter-acting* node and an *intra-acting* node at this point of present explication. Node 'i' is said to be the inter-acting node if it belongs to the set ϕ_k; in contrast, 'i' is said to be an intra-acting node if it belongs to $\tilde{\phi}_k - \phi_k$. These conceptualizations imply that an inter-acting node can receive information from the other network clusters, whereas an intra-acting node can only exchange information within the same network cluster.

Next, suppose the synchronization errors of node 'i' in the k -th network are defined as

$$e_i^k(t) = x_i^k(t) - s^k(t). \tag{13}$$

Then the system model of error dynamics for the multi-network (11) can be derived in the form:

$$\begin{cases} \dot{e}_i^k(t) = f^k(x_i^k(t)) - f^k(s^k(t)) + \sum_{j=1}^{N_k} a_{ij}^{kk} e_j^k(t) \\ \quad + \sum_{l=1,l\ne k}^{r} \sum_{j=1}^{[\alpha N_l]} (m_{ij}^{kl}(t) + a_{ij}^{kl}) \Gamma^{kl} e_j^l(t) \\ \quad + \sum_{l=1,l\ne k}^{r} \sum_{j=[\alpha N_l]+1}^{N_l} \delta_i^{kl}(t) a_{ij}^{kl} \Gamma^{kl} e_j^l(t) + \sum_{j=1}^{N_k} a_{ij}^{kk} s^k(t) \\ \quad + \sum_{l=1,l\ne k}^{r} \sum_{j=1}^{[\alpha N_l]} (m_{ij}^{kl}(t) + a_{ij}^{kl}) \Gamma^{kl} s^l(t) \\ \quad + \sum_{l=1,l\ne k}^{r} \sum_{j=[\alpha N_l]+1}^{N_l} \delta_i^{kl}(t) a_{ij}^{kl} \Gamma^{kl} s^l(t) + u_i^k(t), \\ i = 1,2,\cdots,N_k, k = 1,\ldots r. \end{cases} \tag{14}$$

Lemma 1 [47]: If node 'i' is an intra-acting node, namely, $i \in \tilde{\phi}_k - \phi_k$, then the following results hold:

$$\begin{aligned} &\sum_{k=1}^{r} \sum_{j=1}^{N_k} a_{ij}^{kk} s^k(t) = 0,\ \sum_{l=1,l\ne k}^{r} \sum_{j=1}^{[\alpha N_l]} (m_{ij}^{kl}(t) + a_{ij}^{kl}) \Gamma^{kl} s^l(t) = 0, \\ &\sum_{l=1,l\ne k}^{r} \sum_{j=[\alpha N_l]+1}^{N_l} \delta_i^{kl}(t) a_{ij}^{kl} \Gamma^{kl} s^l(t) = 0, \\ &i = 1,\ldots,N_k, k = 1,\ldots,r. \end{aligned} \tag{15}$$

Lemma 2 [47]: For a symmetric matrix $M = (m_{ij})_{N\times N}$ and $D = \mathrm{diag}(d_1,\cdots,d_q,0,\cdots,0)_{N\times N}$, $i = 1,2,\cdots,q (1 \le q \le N)$, suppose

$$M - D = \begin{bmatrix} A - \tilde{D} & C \\ C^{\mathrm{T}} & M_q \end{bmatrix}, \qquad (16)$$

where M_q is the minor matrix of M obtained by removing its first q row-column pairs, and where:

$A=(a_{ij})_{q\times q}$, $a_{ij}=a_{ji}=m_{ij}$ $i,j=1,2,\ldots,q$, $C=(c_{ij})_{q\times(N-q)}$, $c_{ij}=m_{ij}$, $i=1,2,\ldots,q$, $j=q+1,\ldots,N$ $M_q=(m_{q_{ij}})_{(N-q)\times(N-q)}$, $m_{q_{ij}}=m_{q_{ji}}=m_{i+q,j+q}$, $i,j=1,2\ldots,N-q$.

If $d_i > \lambda_{\max}\left(A - CM_q^{-1}C^{\mathrm{T}}\right)$, then $M-D<0$ is equivalent to $M_q<0$.

Lemma 3 [18,41]: *(Gerschgorin Disc Theorem)* Let matrix $A=(a_{ij})_{n\times n}$ be a complex matrix and let $R_i(A)=\sum_{j=1,j\neq i}^{n}|a_{ij}|, 1\le i\le n$ represent the deleted absolute row sums of A. Then all the eigenvalues of A are located in the union of the following n discs:

$$G(A)=\bigcup_{i=1}^{n}\{z\in C:|z-a_{ii}|\le R_i(A)\}. \qquad (17)$$

Remark 2: If A is a real-valued symmetric matrix, it follows from Lemma 1 that the eigenvalues λ of A satisfy

$$\min_{1\le i\le n}(a_{ii}-R_i(A))\le\lambda\le\max_{1\le i\le n}(a_{ii}+R_i(A)). \qquad (18)$$

Lemma 4 [41,30]: Let assume that A, B are $N\times N$ Hermitian matrices and $\alpha_1\ge\alpha_2\ge\ldots\ge\alpha_N$, $\beta_1\ge\beta_2\ge\ldots\ge\beta_N$, $\gamma_1\ge\gamma_2\ge\ldots\ge\gamma_N$ are the eigenvalues of matrices A, B, and $A+B$, respectively. Then $\alpha_i+\beta_N\le\gamma_i\le\alpha_i+\beta_1$ hold true.

Now the main results can be presented.

Main Innovated Results

A sufficient asymptotic stability condition ought to be derived for the considered multi-networks assuming they may happen to have both permanent fault and recoverable fault in connection-coupled nodes among different networks [32]. Though, the concepts and ideas of Lyapunov stability theory from [25] ought be employed as applied to networks [32,48,49] and synchronization of networks [34,50-53].

Theorem 1: Under the Assumptions 1-5, the controlled multi-network (11) with the controller system (12 a, b) can accomplish the desired synchronization *if* the following inequality

$$\Theta - D + A^s < 0 \qquad (19)$$

is satisfied, where:

$$D=\mathrm{diag}(d_1^1I_{n_1},\ldots,d_{N_1}^1I_{n_1},d_1^2I_{n_2},\ldots,d_{N_2}^2I_{n_2},\ldots,d_1^rI_{n_r},\ldots,d_{N_r}^rI_{n_r});$$

$$\Theta=\mathrm{diag}(\theta_1 I_{N_1\times n_1},\ldots,\theta_r I_{N_r\times n_r}); \quad A^s=\frac{A+A^{\mathrm{T}}}{2};$$

$$A=\begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12} & \cdots & \tilde{A}_{1r}\\ \tilde{A}_{21} & \tilde{A}_{22} & \cdots & \tilde{A}_{2r}\\ \vdots & \vdots & \ddots & \vdots\\ \tilde{A}_{r1} & \tilde{A}_{r2} & \cdots & \tilde{A}_{rr}\end{bmatrix}, \quad \tilde{A}_{kk}=\bar{A}_{kk}\otimes I_{n_k}, \quad \tilde{A}_{kl}=\bar{A}_{kl}\otimes\Gamma^{kl}, \text{ when } j=1,2,\ldots,[\alpha N_l];$$

$(\bar{A}_{kl})_{ij}=\bar{m}_{ij}^{kl}+a_{ij}^{kl}$, when $j=[\alpha N_l]+1,\ldots\ldots,N_l$;

$(\bar{A}_{kl})_{ij}=\bar{\delta}_i^{kl}a_{ij}^{kl}$, $(\bar{A}_{kk})_{ij}=a_{ij}^{kk}$, $i,j=1,2,\ldots,N_k$, $k=1,2,\ldots,r$.

Proof [45]. Finding appropriate composite Lyapunov function is the crucial task of the problem in the synthesis of multi-network systems. In this case Lyapunov function candidate is built by using component Lyapunov-like quadratic forms:

$$V_1(t)=\frac{1}{2}\sum_{k=1}^{r}\sum_{i=1}^{N_k}e_i^k(t)^{\mathrm{T}}e_i^k(t), \qquad (20)$$

$$V_2(t)=\frac{1}{2}\sum_{k=1}^{r}\sum_{i\in\phi_k}\sum_{l=1,l\neq k}^{r}\sum_{j=1}^{[\alpha N_l]}\frac{(m_{ij}^{kl}(t)+a_{ij}^{kl})^2}{\beta}, \qquad (21)$$

$$V_3(t)=\frac{1}{2}\sum_{k=1}^{r}\sum_{i\in\phi_k}\sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l}(h_{ij}^{kl}(t)-\bar{\delta}_i^{kl}a_{ij}^{kl})^2, \qquad (22)$$

$$V_4(t)=\frac{1}{2}\sum_{k=1}^{r}\sum_{i\in\phi_k}\sum_{j=1}^{N_k}(h_{ij}^{kk}(t)-a_{ij}^{kk})^2. \qquad (23)$$

Thus, composite Lyapunov function candidate is given as

$$V(t)=V_1(t)+V_2(t)+V_3(t)+V_4(t). \qquad (24)$$

Next, the differential operator of the first time-derivative of Lyapunov function (20), conveniently denoted as $\ell V_1(t)$, is calculated. Due to Assumptions 1 and 2 as well as Lemma 1, and supported by Lemmas 3 and 4, one can calculate:

$$\begin{aligned}
\ell V_1(t) &= \sum_{k=1}^{r}\sum_{i=1}^{N_k}e_i^k(t)^{\mathrm{T}}\dot{e}_i^k(t)\\
&= \sum_{k=1}^{r}\sum_{i\in\phi_k}e_i^k(t)^{\mathrm{T}}[f^k(x_i^k(t))-f^k(s^k(t))+\sum_{j=1}^{N_k}a_{ij}^{kk}e_j^k(t)\\
&+\sum_{l=1,l\neq k}^{r}\sum_{j=1}^{[\alpha N_l]}(m_{ij}^{kl}(t)+a_{ij}^{kl})\Gamma^{kl}e_j^l(t)+\sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l}\delta_i^{kl}(t)a_{ij}^{kl}\Gamma^{kl}e_j^l(t)+\sum_{j=1}^{N_k}a_{ij}^{kk}s^k(t)\\
&+\sum_{l=1,l\neq k}^{r}\sum_{j=1}^{[\alpha N_l]}(m_{ij}^{kl}(t)+a_{ij}^{kl})\Gamma^{kl}s^l(t)-\sum_{j=1}^{N_k}h_{ij}^{kk}(t)s^k(t)-d_i^ke_i^k(t)]\\
&+\sum_{k=1}^{r}\sum_{i\in\phi_k}e_i^k(t)^{\mathrm{T}}\left[\sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l}\delta_i^{kl}(t)a_{ij}^{kl}\Gamma^{kl}s^l(t)-\sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l}h_{ij}^{kl}\Gamma^{kl}s^l(t)\right]\\
&+\sum_{k=1}^{r}\sum_{i\in\tilde{\phi}_k-\phi_k}e_i^k(t)^{\mathrm{T}}\left[f^k(x_i^k(t))-f^k(s^k(t))+\sum_{j=1}^{N_k}a_{ij}^{kk}e_j^k(t)-d_i^ke_i^k(t)\right]
\end{aligned}$$

$$\begin{aligned}
&\le\sum_{k=1}^{r}\sum_{i\in\phi_k}e_i^k(t)^{\mathrm{T}}\left[(\theta_k-d_i^k)e_i^k(t)+\sum_{j=1}^{N_k}a_{ij}^{kk}e_j^k(t)+\sum_{l=1,l\neq k}^{r}\sum_{j=1}^{[\alpha N_l]}(m_{ij}^{kl}(t)+a_{ij}^{kl})\Gamma^{kl}e_j^l(t)\right.\\
&\left.+\sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l}\delta_i^{kl}(t)a_{ij}^{kl}\Gamma^{kl}e_j^l(t)\right]\\
&+\sum_{k=1}^{r}\sum_{i\in\phi_k}e_i^k(t)^{\mathrm{T}}\left[\sum_{j=1}^{N_k}a_{ij}^{kk}s^k(t)+\sum_{l=1,l\neq k}^{r}\sum_{j=1}^{[\alpha N_l]}(m_{ij}^{kl}(t)+a_{ij}^{kl})\Gamma^{kl}s^l(t)-\sum_{j=1}^{N_k}h_{ij}^{kk}(t)s^k(t)\right]\\
&+\sum_{k=1}^{r}\sum_{i\in\phi_k}e_i^k(t)^{\mathrm{T}}\left[\sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l}\delta_i^{kl}(t)a_{ij}^{kl}\Gamma^{kl}s^l(t)-\sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l}h_{ij}^{kl}\Gamma^{kl}s^l(t)\right]\\
&+\sum_{k=1}^{r}\sum_{i\in\tilde{\phi}_k-\phi_k}e_i^k(t)^{\mathrm{T}}\left[(\theta_k-d_i^k)e_i^k(t)+\sum_{j=1}^{N_k}a_{ij}^{kk}e_j^k(t)\right].
\end{aligned} \qquad (25)$$

Similarly, by calculating differential operators of the first time-derivative of (21)-(23), one can establish results:

$$\ell V_2(t)=-\sum_{k=1}^{r}\sum_{i\in\phi_k}\sum_{l=1,l\neq k}^{r}\sum_{j=1}^{[\alpha N_l]}(m_{ij}^{kl}(t)+a_{ij}^{kl})e_i^k(t)^{\mathrm{T}}\Gamma^{kl}s^l(t), \qquad (26)$$

$$\ell V_3(t)=\sum_{k=1}^{r}\sum_{i\in\phi_k}\sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l}(h_{ij}^{kl}(t)-\bar{\delta}_i^{kl}a_{ij}^{kl})e_i^k(t)^{\mathrm{T}}\Gamma^{kl}s^l(t), \qquad (27)$$

$$\ell V_4(t)=\sum_{k=1}^{r}\sum_{i\in\phi_k}\sum_{j=1}^{N_k}(h_{ij}^{kk}(t)-a_{ij}^{kk})e_i^k(t)^{\mathrm{T}}s^k(t). \qquad (28)$$

Next, the combined consideration of the above four results yields the first time-derivative of the composite Lyapunov function, conceptualized by means of (25)-(28), as follows:

$$\ell V(t) = \ell V_1(t) + \ell V_2(t) + \ell V_3(t) + \ell V_4(t). \quad (29)$$

Hence the following end result is readily found as follows:

$$\begin{aligned}\ell V(t) = &\sum_{k=1}^{r}\sum_{i\in\phi_k} e_i^k(t)^{\mathrm{T}}\Bigg[(\theta_k - d_i^k)e_i^k(t) + \sum_{j=1}^{N_k} a_{ij}^{kk} e_j^k(t)\\ &+ \sum_{l=1,l\neq k}^{r}\sum_{j=1}^{[\alpha N_l]}(m_{ij}^{kl}(t)+a_{ij}^{kl})\Gamma^{kl}e_j^l(t) + \sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l}\delta_i^{kl}(t)a_{ij}^{kl}\Gamma^{kl}e_j^l(t)\Bigg]\\ &+\sum_{k=1}^{r}\sum_{i\in\phi_k} e_i^k(t)^{\mathrm{T}}\Bigg[\sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l}\delta_i^{kl}(t)a_{ij}^{kl}\Gamma^{kl}s^l(t) - \sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l}\bar{\delta}_i^{kl}a_{ij}^{kl}\Gamma^{kl}s^l(t)\Bigg]\\ &+\sum_{k=1}^{r}\sum_{i\in\tilde{\phi}_k-\phi_k} e_i^k(t)^{\mathrm{T}}\Bigg[(\theta_k - d_i^k)e_i^k(t) + \sum_{j=1}^{N_k} a_{ij}^{kk} e_j^k(t)\Bigg].\end{aligned} \quad (30)$$

The calculation of the expected value of (30) does yield:

$$\begin{aligned}E\{\ell V(t)\} = &\sum_{k=1}^{r}\sum_{i\in\phi_k} e_i^k(t)^{\mathrm{T}}\Bigg[(\theta_k - d_i^k)e_i^k(t) + \sum_{j=1}^{N_k} a_{ij}^{kk} e_j^k(t)\\ &+ \sum_{l=1,l\neq k}^{r}\sum_{j=1}^{[\alpha N_l]}(m_{ij}^{kl}(t)+a_{ij}^{kl})\Gamma^{kl}e_j^l(t) + \sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l}\bar{\delta}_i^{kl}a_{ij}^{kl}\Gamma^{kl}e_j^l(t)\Bigg]\\ &+\sum_{k=1}^{r}\sum_{i\in\tilde{\phi}_k-\phi_k} e_i^k(t)^{\mathrm{T}}\Bigg[(\theta_k - d_i^k)e_i^k(t) + \sum_{j=1}^{N_k} a_{ij}^{kk} e_j^k(t)\Bigg]\end{aligned} \quad (31)$$

Thus, by virtue of Assumption 2, one can establish

$$\begin{aligned}&\sum_{k=1}^{r}\sum_{i\in\phi_k} e_i^k(t)^{\mathrm{T}}[(\theta_k - d_i^k)e_i^k(t) + \sum_{j=1}^{N_k} a_{ij}^{kk} e_j^k(t)\\ &+ \sum_{l=1,l\neq k}^{r}\sum_{j=1}^{[\alpha N_l]}(m_{ij}^{kl}(t)+a_{ij}^{kl})\Gamma^{kl}e_j^l(t) + \sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l}\bar{\delta}_i^{kl}a_{ij}^{kl}\Gamma^{kl}e_j^l(t)]\\ &\leq \sum_{k=1}^{r}\sum_{i\in\phi_k}^{N_k} e_i^k(t)^{\mathrm{T}}\left[(\theta_k - d_i^k)e_i^k(t)\right] + \sum_{k=1}^{r}\sum_{i\in\phi_k}^{N_k} e_i^k(t)^{\mathrm{T}}\sum_{j=1}^{N_k} a_{ij}^{kk} e_j^k(t)\\ &+\sum_{k=1}^{r}\sum_{i\in\phi_k}^{N_k}\sum_{l=1,l\neq k}^{r}\sum_{j=1}^{[\alpha N_l]} |e_i^k(t)|^{\mathrm{T}}(\bar{m}_{ij}^{kl}+a_{ij}^{kl})\Gamma^{kl}|e_j^l(t)|\\ &+\sum_{k=1}^{r}\sum_{i\in\phi_k}^{N_k}\sum_{l=1,l\neq k}^{r}\sum_{j=[\alpha N_l]+1}^{N_l} e_i^k(t)^{\mathrm{T}}\bar{\delta}_i^{kl}a_{ij}^{kl}\Gamma^{kl}e_j^l(t),\end{aligned} \quad (32)$$

where $|e_i^k(t)| = (|e_{i1}^k(t)|, |e_{i2}^k(t)|, \cdots, |e_{in_k}^k(t)|)^{\mathrm{T}} \in \mathbb{R}^{n_k}$. Then combining results (30) and (31) does yield

$$E\{\ell V(t)\} \leq |e(t)|^{\mathrm{T}}(\Theta - D + A^s)|e(t)| \quad (33)$$

where:

$$|e(t)| = (|e_1^1(t)|^{\mathrm{T}}, \ldots, |e_{N_1}^1(t)|^{\mathrm{T}}, |e_1^2(t)|^{T}, \ldots,$$
$$|e_{N_2}^2(t)|^{\mathrm{T}}, \ldots, |e_1^r(t)|^{\mathrm{T}}, \ldots, |e_{N_r}^r(t)|^{\mathrm{T}})^{\mathrm{T}}.$$

It is therefore that, whenever inequality (19) is satisfied, then the expectation the first time-derivative of Lyapunov function is $E\{\ell V(t)\} < 0$, that is negative definite, which implies asymptotic stability of the error dynamics of the controlled network system. Thus the synchronization in the mean-square sense for the multi-network (11) under the pinning control (12 a, b) is guaranteed asymptotically stable. □

Remark 3: It should be noted, whenever inequality (19) is satisfied, so are satisfied inequalities implied by Lemmas 2 and 3 as well as Remark 2.

Theorem 1 guaranteed the synchronization in general multi-network structure is feasible by employing the pinning control (12 a, b). Therefore, given the validity of the adopted Lemmas 1-4, one can conclude: (i) At least one design algorithm for the gains of pinning control (12 a) is feasible too hence it can be found; and (ii) The pinning control (12 a, b), provided it does exist, then it does govern the multi-network (4) or the network-of-networks systemic structure (1) by operating as an acting supervisory coordinator [38, 41].

One such design algorithm via network node manipulation, in the view of right-hand matrices in the inequality (19), has been derived as follows [25,42]. For this purpose, the nodes taken are re-ordered as outlined below: The inter-acting nodes that are subject to control are adopted as the first nodes, while the remaining nodes are left uncontrolled. In this way a control selection is designed. At the same time the matrix

$$G = (g_{ij})_{N\times N} = \Theta + A^s \quad (34)$$

is constructed. Therefore, instead of equality (15) in Lemma 2, the following result is obtained:

$$G - \tilde{D} = \begin{bmatrix} B-\tilde{D} & C \\ C^{\mathrm{T}} & G_q \end{bmatrix}. \quad (35)$$

In here quantities denote: $B = (b_{ij})_{q\times q}, b_{ij} = b_{ji} = g_{ij}, i = 1,\ldots,q$, gains $\tilde{D} = \mathrm{diag}(d_1, \cdots, d_q)$, $C = (c_{ij})_{q\times(N-q)}$, $c_{ij} = g_{ij}$, $i = 1,\ldots,q,$, $j = q+1, q+2, \ldots, N$. In (35) G_q is the minor of matrix G by removing its first q row-column pairs.

Next, the effect influence of the control node selection is discussed. For this purpose, let it be defined and denoted:

$$-a_{ii}^{kk} = \sum_{j=1,j\neq i}^{N_k} a_{ij}^{kk} + \sum_{l=1,l\neq k}^{r}\sum_{j=1}^{N_l} a_{ij}^{kl} = \mathrm{Degin}(k_i), \quad (36)$$

$$\sum_{j=1,j\neq i}^{N_k} a_{ji}^{kk} + \sum_{l=1,l\neq k}^{r}\sum_{j=1}^{N_l} a_{ji}^{kl} = \mathrm{Degout}(k_i). \quad (37)$$

In here, $\mathrm{Degin}(k_i)$ and $\mathrm{Degout}(k_i)$ denote the in-degree and the out-degree of a given node vertex 'i', respectively.

In order to ensure G is negative definite always, Eq. (34) must yield diagonal elements less than zero. Theorem 1 still remains satisfied if the nodes whose diagonal elements are bigger than zero are being kept as controlled ones. Matrix has diagonal elements determined by equation

$$G_{ii}^k = \theta_k + a_{ii}^{kk} = \theta_k - \mathrm{Degin}(k_i) \quad (38)$$

Note that the smaller the in-degrees of a vertex is, then the more easily the diagonal element may be greater than zero. Thus pining-control candidate nodes ought to be selected among some vertices with very small in-degree values. It should be noted though, some vertices with very small in-degrees are also bound to receive very little information from all the other vertices; therefore it is unlikely these be readily controlled by other nodes. Thus it is necessary to apply additional control effort with other nodes in order to achieve desired controlled synchronization of the considered network.

From Lemma 2, whenever (19) is satisfied, if (37) and (38) are also satisfied

$$G_q < 0 \text{ ,} \qquad (39)$$

$$d_i > \lambda_{\max}\left(B - CG_q^{-1}C^{T}\right) \text{ ,} \quad (40)$$

then the multi-network (11) is globally synchronized under the given pinning controllers (12 a. b).

By virtue of Lemma 3 and Remark 2, one can readily establish the arbitrary eigenvalue $\lambda \le \max(G_{q_{ii}} + \sum_{j=1, j\ne i}^{N} G_{q_{ij}})$ of the matrix G is such that λ is less than the largest row sum of G_q. Thus the smaller the largest row sum of G_q is, the matrix is more likely to be negative definite when the largest row sum of the matrix G_q is small. The matrix G_q is more likely to be negative definite when there are subject to control all the nodes for which the row-sum of the matrix G nodes has greater value. By means of (34) and (35), one finds that the row-sum of matrix G is given as follows:

$$\theta_k + \frac{\text{Degout}(k_i) - \text{Degin}(k_i)}{2} \qquad 39\ (31)$$

It is also apparent, the greater the difference out-degree and in-degree for a given node is, the more likely the row-sum of the matrix G is greater than zero.

In turn, it is readily concluded the network is more likely to reach synchronization if the vertices with large out-degrees are being pinned for controller infrastructure. For, the vertices with large out-degrees can influence many other vertices in the multi-network structure. These findings yield the below summarized algorithm for the pinned-node design selection scheme.

Pinned-Node Selection Scheme Algorithm

1) All inter-acting nodes are controlled and the total number is denoted as q_0.

2) In the remaining nodes, the nodes with diagonal elements of matrix G that are greater than zero are controlled via the remaining nodes, and the total number is denoted as q_1.

3) All remaining nodes are sorted by the row sum of matrix descending order, let $q = q_0 + q_1$.

4) Calculate whether the matrix G_q is negative definite. If G_q is not negative definite, let $q = q + 1$, go to step (3); otherwise, go to calculate feedback control gains by using inequality (40) via maximal eigenvalue of resulting matrix in the right-hand side; *Stop*.

A Case-Study Example and Simulation Results

A complex systemic structure possessing three directional networks, namely **r =3**, is taken as an example for numerical simulation in order to clearly illustrate what functioning the above presented theoretical results imply. The network-of-networks structure is shown in Figure 1. The number of nodes in each network is $N_1=4$, $N_2=3$, $N_3=3$, respectively (Figure 1).

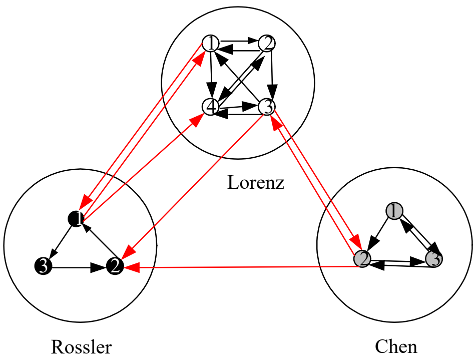

Figure 1: Topological structure of the benchmark multi-network.

The nodes in the first network are selected to represent the four-dimensional hyper-chaotic Lorenz system. The nodes in the second network are chosen to construct three-dimensional Rossler system. The nodes in the third network are selected to create a three-dimensional Chen system [45]. The respective mathematical models are presented further below.

Nodes absorbing hyper-chaotic Lorenz dynamic system are described by equations:

$$\begin{aligned}
\dot{x}_1(t) &= 10(x_2(t) - x_1(t)) + x_4(t),\\
\dot{x}_2(t) &= 28x_1(t) - x_2(t) - x_1(t)x_3(t),\\
\dot{x}_3(t) &= x_1(t)x_2(t) - 8/3x_3(t),\\
\dot{x}_4(t) &= -x_1(t)x_3(t) + 13/10x_3(t).
\end{aligned}$$

Nodes absorbing Rossler dynamic system are described by equations:

$$\begin{aligned}
\dot{x}_1(t) &= -x_2(t) - x_3(t),\\
\dot{x}_2(t) &= x_1(t) + 0.2x_2(t),\\
\dot{x}_3(t) &= x_1(t)x_3(t) - 5.7x_3(t) + 0.2.
\end{aligned}$$

Nodes absorbing Chen dynamic system are described by equations:

$$\begin{aligned}
\dot{x}_1(t) &= 35(x_2(t) - x_1(t)),\\
\dot{x}_2(t) &= -7x_1(t) + 28x_2(t) - x_1(t)x_3(t),\\
\dot{x}_3(t) &= -3x_3(t) + x_1(t)x_2(t).
\end{aligned}$$

Notice that chaotic attractors of the hyper-chaotic Lorenz system, Rossler system and Chen system appear with bounded regions hence numerical simulations by computer are feasible. Thus, by means of numerical simulations, one can find that there exist some constants for the first network as follows

$$M_{11} = 25, M_{12} = 25, M_{13} = 45,\ M_{14} = 180,$$

such that $|s_{11}| \le M_{11}, |s_{12}| \le M_{12}, |s_{13}| \le M_{13}, |s_{14}| \le M_{14}$.

For the parameters of Assumption 1, they can be calculated using the following method as illustrated with its application to o the first network as a guidance example. Indeed, one has:

$$
\begin{aligned}
&(x_i - s^1)^{\mathrm{T}}(f^1(x_i) - f^1((s^1)) \\
&= e_i^{\mathrm{T}}(10e_{i2} - 10e_{i1} + e_{i4}, 28e_{i1} - e_{i2} - x_{i1}x_{i3} + s_{11}s_{13}, \\
&x_{i1}x_{i2} - s_{11}s_{12} - 8/3e_{i3}, 1.3e_{i4} - x_{i1}x_{i3} + s_{11}s_{13})^{\mathrm{T}} \\
&\le -10e_{i1}^{\ 2} - e_{i2}^{\ 2} - 8/3e_{i3}^{\ 2} + 1.3e_{i4}^{\ 2} + (38 + M_{13})|e_{i1}e_{i2}| + M_{12}|e_{i1}e_{i3}| \\
&\quad + (1 + M_{13})|e_{i1}e_{i4}| + M_{11}|e_{i3}e_{i4}| \\
&\le (-10 + \frac{\varepsilon_1(38 + M_{13})}{2} + \frac{\varepsilon_2 M_{12}}{2} + \frac{\varepsilon_3(1 + M_{13})}{2})e_{i1}^{\ 2} + (-1 + \frac{38 + M_{13}}{2\varepsilon_1})e_{i2}^{\ 2} \\
&\quad + (-\frac{8}{3} + \frac{M_{12}}{2\varepsilon_2} + \frac{\varepsilon_4 M_{11}}{2})e_{i3}^{\ 2} + (1.3 + \frac{1 + M_{13}}{2\varepsilon_3} + \frac{M_{11}}{2\varepsilon_4})e_{i4}^{\ 2}
\end{aligned}
$$

where $\varepsilon_i (i = 1,2,3,4)$ are arbitrary positive constants. Then computing using $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (1.0, 0.4, 0.6, 1.2)$ yields $\theta_1 = 50.3$ hence Assumption 1 holds true. For the second and third networks it can be found $\theta_2 = 38.1$, $\theta_2 = 44.5$.

The coupling matrix of the multi-network is given as:

$$
A = \left[\begin{array}{cccc:ccc:ccc}
-100 & 50 & 0 & 40 & 10 & 0 & 0 & 0 & 0 & 0 \\
30 & -110 & 40 & 40 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 0 & -140 & 80 & 0 & 10 & 0 & 0 & 30 & 0 \\
0 & 40 & 30 & -70 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
10 & 0 & 0 & 30 & -70 & 0 & 30 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 70 & -70 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 50 & -50 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -40 & 30 & 10 \\
0 & 0 & 20 & 0 & 0 & 10 & 0 & 0 & -60 & 30 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 70 & 20 & -90
\end{array}\right]
$$

Initial values for the hyper-chaotic Lorenz system, for the Rossler system, and for the Chen system, respectively, are $x_0 = (7,6,5,4)^{\mathrm{T}}$, $x_0 = (-10,0,10)^{\mathrm{T}}$ and $x_0 = (-4,2,8)^{\mathrm{T}}$. In turn, computer simulations yielded the time-evolution curves of state response that are depicted in Figures 2-4.

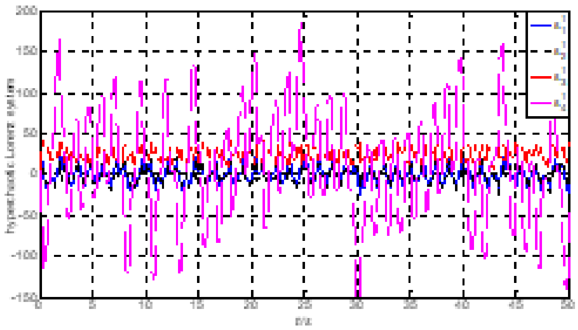

Figure 2: State responses of the hyper-chaotic Lorenz system in the time domain when $s_{10} = (7,6,5,4)^{\mathrm{T}}$.

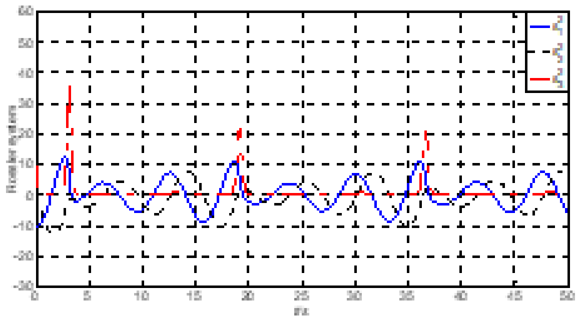

Figure 3: State responses of the Rossler system in the time domain when $s_{20} = (-10,0,10)^{\mathrm{T}}$.

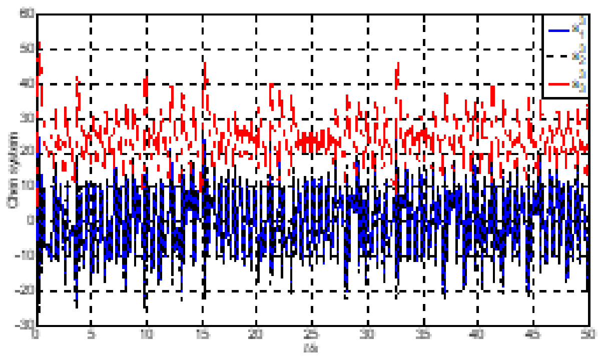

Figure 4: State responses of the Chen system in the time domain when $s_{30} = (-4,2,8)^{\mathrm{T}}$.

The hyper-chaotic Lorenz system has the state response shown in Figure 2, the Rossler system has the state response shown in Figure 3, and the Chen system has the state response shown in Figure 4. They all are the final synchronization states of three directional networks respectively.

Simulation is carried out without applying control to compare the effect of multi-network synchronization before and after implanting the control infrastructure. The error evolution curves in the time domain for the nodes in multi-network system and for isolated nodes are shown in Figures 5-7. As it can be seen from these figures, each node in the multi-network cannot achieve synchronization only on the grounds of the coupling effects among the networks (Figure 5-7).

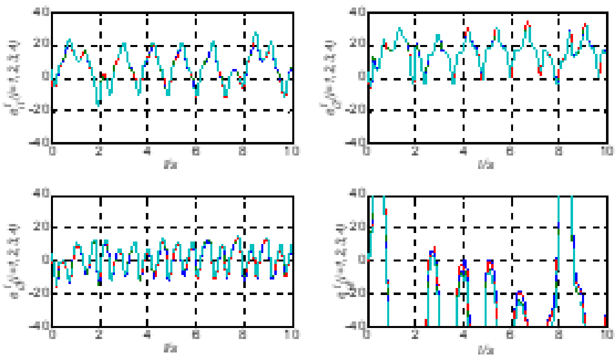

Figure 5: Error response curve $e_i^1 \in \mathbb{R}^4, i = 1,2,3,4$ in the time domain of the first uncontrolled network.

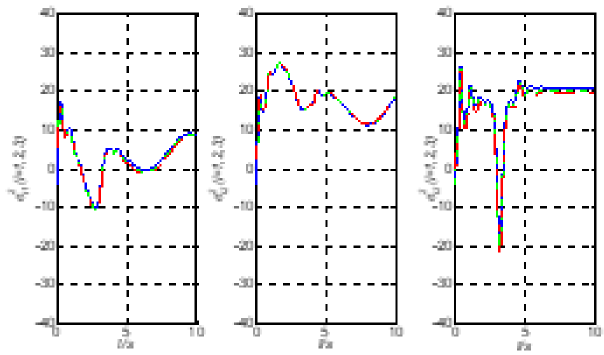

Figure 6: Error response curve $e_i^2 \in \mathbb{R}^3, i = 1,2,3$ in the time domain of the second uncontrolled network.

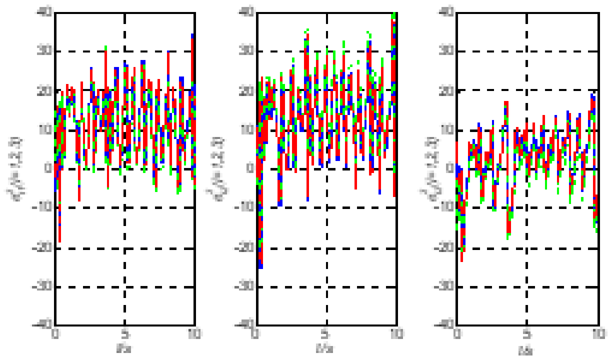

Figure 7: Error response curve $e_i^3 \in \mathbb{R}^3, i = 1,2,3$ in the time domain of the third uncontrolled network.

Results Under Specifically Designed Pinning Scheme: Let consider the case where a permanent fault and a recoverable fault occur among different networks; step signal is the permanent one. By choosing $m_{11}^{12}(t) = 40*\varepsilon(t-3)$, $m_{11}^{21}(t) = 35*\varepsilon(t-3)$, $m_{22}^{32}(t) = 30*\varepsilon(t-3)$, $m_{32}^{12}(t) = 35*\varepsilon(t-3)$. The fault curve is shown in Figure 8. The probabilities of the normal connection of recoverable nodes are $\bar{\delta}_3^{13} = 0.4$, $\bar{\delta}_1^{21} = 0.75$. Recoverable fault curves are shown in Figure 9 and Figure 10 (Figure 8-10).

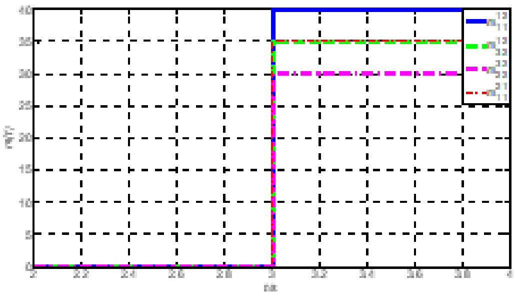

Figure 8: Evolution curve of the permanent fault in time-domain.

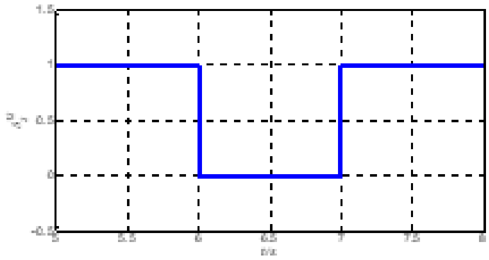

Figure 9: Evolution curve of the recoverable fault in time-domain: Recoverable failure happens about the third node in the first controlled network.

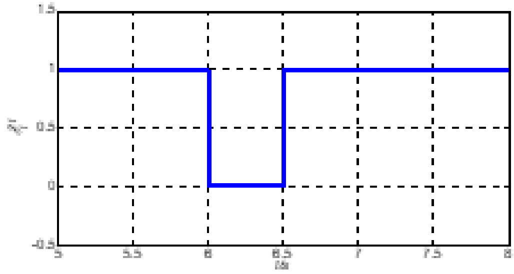

Figure 10: Evolution curve of the recoverable fault in time-domain: Recoverable failure happens about the first node in the second controlled network.

Substituting the simulation values into (2.24) and (2.25), and the nodes are selected according to the node selection scheme:

(1) The inter-acting nodes 1 and 3 in the first network, the inter-acting node 1 in the second network and the inter-acting node 2 in the third network are controlled. The total number of inter-act nodes is denoted as $q_0 = 4$.

(2) In the remaining nodes, the node that diagonal element of the matrix G is more than zero is controlled, which is the node 1 in the third network. The number of nodes is $q_1 = 1$.

(3) All remaining nodes are sorted by the row sum of matrix descending order, let $q = q_0 + q_1$.

(4) G_q is negative definite through calculating when the nodes 1, 3 in the first network, the node 1 in the second network and the nodes 1,2 in the third network are controlled. Theorem 1 is satisfied by (30) to calculate feedback control gains, that are designed as follows:

(5) $\begin{cases} d_i^1 = 50, i = 1,3, \\ d_i^1 = 0, i = 2,4; \end{cases}$, $\begin{cases} d_i^2 = 200, i = 1, \\ d_i^2 = 0, i = 2,3; \end{cases}$,and $\begin{cases} d_i^3 = 80, i = 1,2, \\ d_i^2 = 0, i = 3. \end{cases}$

Under the control of the above-mentioned pinning scheme, the permanent fault occurs in the coupling connection among the networks at the third second of time, and the recoverability fault occurs in the coupling connection among the networks at the sixth time. As a result of external interference caused some nodes among networks to disconnect at the same time, after a period of self-adjustment, the node to restore network connectivity, this process alternately occurs randomly. The error response curves are shown in Figures 11-13.

The role of the pinning controller and its impact on the multi-network structure can be clearly from the error response curves in the time domain that are depicted in Figures 11-13 (Figure 11,12).

In the first network, in the absence of failure before/at the nodes, there is needed about 1 second to achieve the synchronization. When the permanent failure occurred in about the first 3 seconds, then about the first 5 seconds are needed to achieve synchronization again. When the recoverable fault occurred in about the first 6 seconds, then about 8 seconds are needed to reach synchronization again. In the second network, in the absence of failure before/at the nodes, about 2 seconds are needed to achieve synchronization (Figure 13).

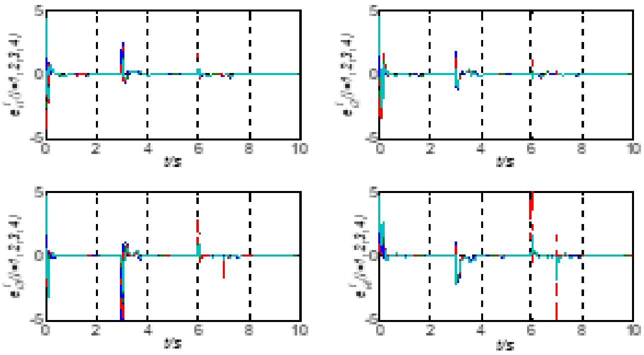

Figure 11: Evolution of the error response curve $e_i^1 \in \mathbb{R}^4, i = 1,2,3,4$ in the time domain for the first controlled network under the special pinning control.

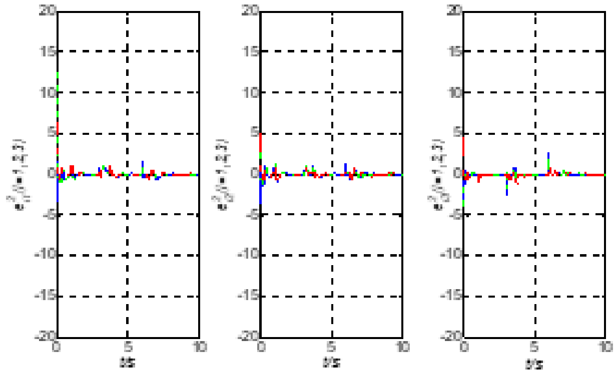

Figure 12: Evolution of the error response curve $e_i^2 \in \mathbb{R}^3, i = 1,2,3$ in the time domain for the second controlled network under the special pinning control .

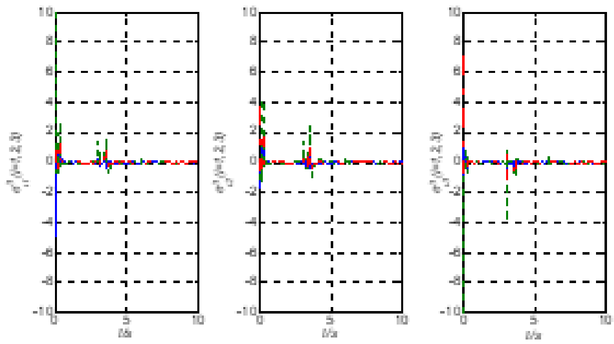

Figure 13: Evolution of the error response curve $e_i^3 \in \mathbb{R}^3, i = 1,2,3$ in the time domain for the third controlled network under the special pinning control.

If and when the permanent failure occurred at the first 3 seconds, then about the first 5 seconds are needed to achieve synchronization again. If the recoverable fault occurred in about the first 6 seconds, then about 8 seconds ere needed to reach the synchronization again. In the third network, when there is no failure before/at the nodes, then about 2.5 seconds are needed in order to achieve the synchronization. When the permanent failure occurred in about the first 3 seconds, then about the 5 first seconds are needed in order to achieve synchronization again. If the recoverable fault occurred in the first 6 seconds, then about 7.5 seconds are needed to reach the synchronization again.

Simulation Results Under Randomly Pinning Control Scheme: When the network topology and the coupling strength are unchanged, the nodes 3,4 in the first network ,the node 2 in the second network and the nodes 2,3 in the third network are not selected as the pinned nodes not in accordance with pinning scheme. There is no feasible solution. The state error curves of nodes in the networks are shown in Figures 14-16. As can be seen from the figures, randomly pinning scheme cannot guarantee synchronization (Figure 14).

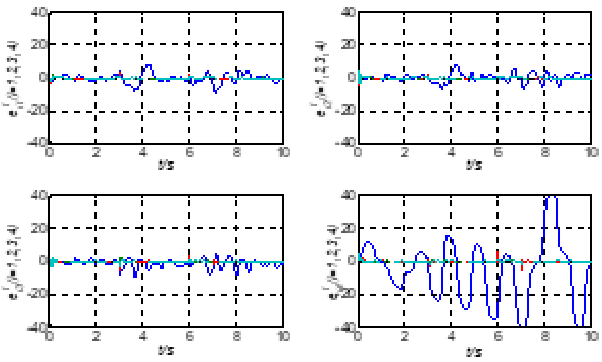

Figure 14: Evolution of the error response curve $e_i^1 \in \mathbb{R}^4, i = 1,2,3,4$ in the time domain for the first controlled network under random pinning control.

When randomly selecting nodes 1, 3 and 4 in the first network, node 1in the second network and node 2 in the third network as the pinned nodes, the controller feedback gains are $d_i^1 = 60, i = 1,3,4$, $d_i^2 = 220, i = 1$, $d_i^3 = 100, i = 2$ by the LMI (Figure 15).

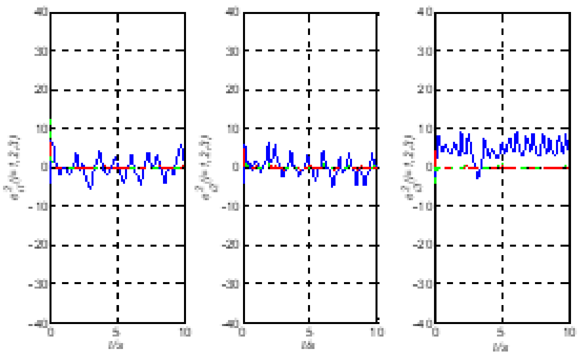

Figure 15: Evolution of the error response curve $e_i^2 \in \mathbb{R}^3, i = 1,2,3$ in the time domain for the second controlled network under random pinning control.

The error evolution curves in the time domain are shown in Figures 17-19. It can be seen from the figures that the first network reaches the synchronization in 8.5 seconds, the second network reaches the synchronization in 9 seconds and the third network reaches the synchronization in 9 seconds under randomly pinning control. Therefore, it takes longer to reach the network synchronization again than in the case of specifically designed pinning control. In addition, the controller feedback gains have greater values, and the impact of the faults on the network is more obvious (Figure 16-19).

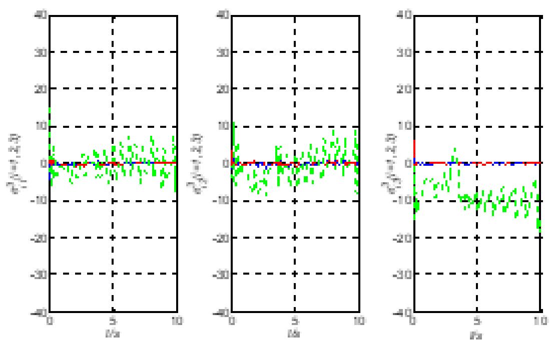

Figure 16: Evolution of the error response curve $e_i^3 \in \mathbb{R}^3, i = 1,2,3$ in the time domain for the third controlled network under random pinning control.

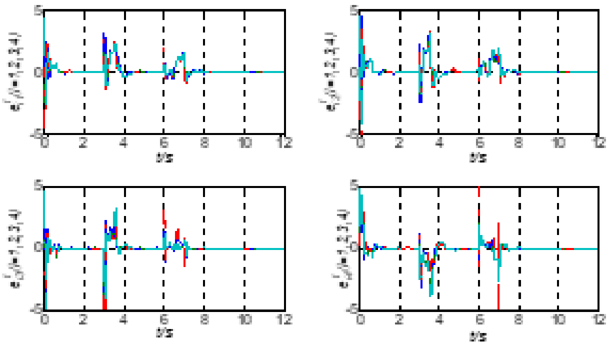

Figure 17: Error response curve $e_i^1 \in \mathbb{R}^4, i = 1,2,3,4$ of the first controlled network with random pinning.

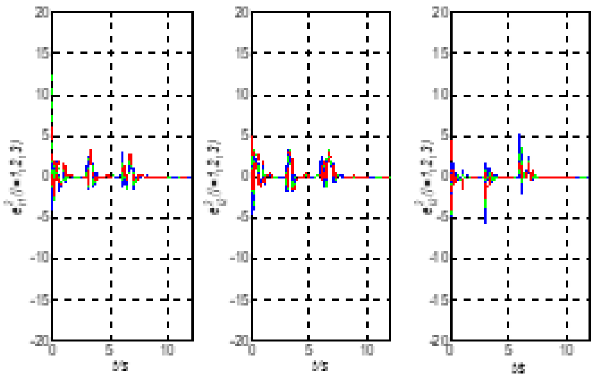

Figure 18: Error response curve $e_i^2 \in \mathbb{R}^3, i = 1,2,3$ of the second controlled network with random pinning.

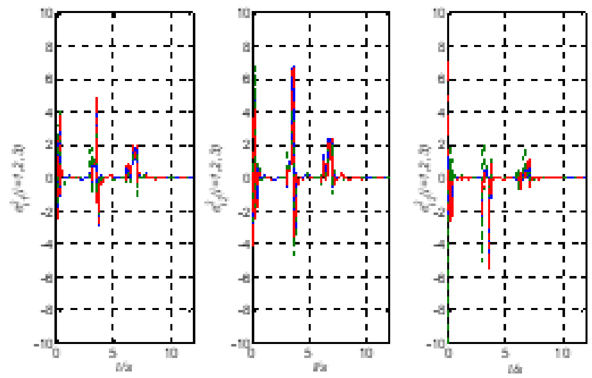

Figure 19: Error response curve $e_i^3 \in \mathbb{R}^3, i = 1,2,3$ of the third controlled network with random pinning.

A simple comparison analysis of Figure 11 thru Figure 13 with Figure 14 thru Figure 19 demonstrates clearly: The specifically designed pinning scheme outperforms considerably the randomly pinning scheme when pinning control reinforces synchronization in a rather complex multi-network structure, involving hyper-chaotic Lorenz, Rossler and Chen node dynamic systems, for the case with the same topologies and coupling strengths as well as the same number of nodes.

Conclusion

In this exploratory research, we have considered the case when both permanent and recoverable failures in inter-network coupling connection edge may occur. A Bernoulli distribution variable is employed to describe changes between the failure and recovery events. On the grounds of proposed pinning control scheme, a feedback controller was designed and applied to some pinned nodes, and a sufficient condition for multi-network synchronization was derived. Finally, vast computer simulation experiments is carried out on a rather complex multi-network consisting of three directed networks. The first selected network was the 4-dimensional hyper-chaotic Lorenz node-systems, the second network was selected as the 3-dimensional Rossler node-systems, and the third network was the 3-dimensional Chen node-systems. The simulation results have demonstrated that the proposed controller can reinforce the synchronization in the multiple network structures in the presence of both permanent and recoverable faults as predicted with the theoretical analysis.

One of the future research topics is envisaged towards an alternative scheme [15,20] of controlled synchronization of multi-networks in presence of both permanent and recoverable faults. The other one worth exploring in such a setting as the above presented solution ideas is the controlled consensus in multi-agent systems with misunderstandings [54], since the consensus problem is not entirely the same as a the controlled synchronization of networks of networks when faults are taken into consideration. The third research topic, and the most difficult one, is to extend recent Chen's results [46] on pinning controllability of complex dynamical networks to the category of complex dynamical multi-networks.

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Conflict of Interest

None.

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